Legendrian Non-simple Two-bridge Knots

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Preliminaries

Contact structure on an oriented 3-manifold M: a completely non-integrable plane field ξ in the tangent bundle of M. Locally, ξ can be defined as the kernel of a 1-form α.
The **standard contact structure** on $\mathbb{R}^3$: the plane field ξ_{std} described as the kernel of $\alpha = dz - ydx$.

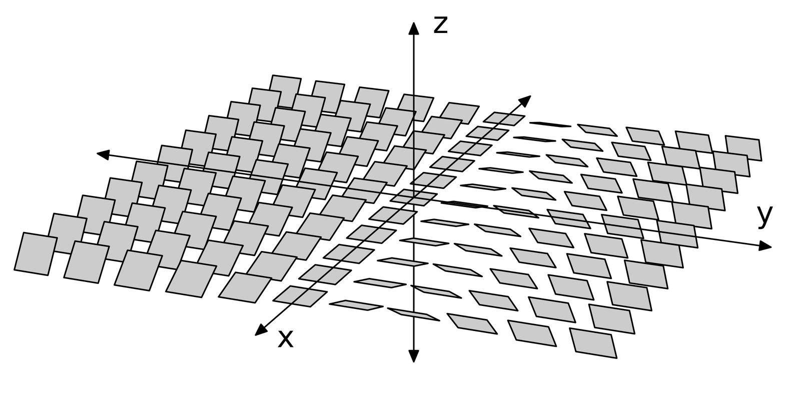

Fig. 1: The standard contact structure [5]

Darboux's theorem: Every contact structure on any 3-manifold locally looks like the standard contact structure on $\mathbb{R}^3$.
An oriented, embedded surface Σ in a contact 3-manifold is called a **convex surface** if there exists a vector field transverse to Σ whose flow preserves the contact structure.
A **Legendrian knot** L in a contact manifold (M^3, ξ) is an embedded S^1 such that $T_xL \subset \xi_x$ for every $x \in L$.
Similarly, a **transverse knot** T in a contact manifold (M^3, ξ) is an embedded S^1 such that $T_xT \pitchfork \xi_x$ for every $x \in T$.
We usually consider the **front projection** of Legendrian knots in $(\mathbb{R}^3, \xi_{std})$, which is their projection to the xz-plane.
Special properties of the front projection:

- no vertical tangencies,
- cusp singularities,
- at every crossing, the arc with smaller slope is over the other.

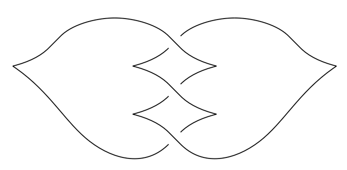
Fig. 2: Front projection of a Legendrian trefoil knot

From its front projection, the original Legendrian knot can be recovered uniquely.
Besides the underlying topological knot type, Legendrian knots have two main classical invariants:

- the **Thurston-Bennequin number**: $\operatorname{tb}(L) = \operatorname{wr}(L) - \frac{1}{2}\#\text{cusps}$, where wr is the **writhe** of the knot;
- the **rotation number**: $\operatorname{rot}(L) = \frac{1}{2}(c_d - c_u)$, where c_d and c_u denote the number of down and up cusps.

The main classical invariant of transverse knots is:

- the **self-linking number**, denoted by sl.

Problem

Two Legendrian knots are **Legendrian isotopic** if they are isotopic through Legendrian knots.
A knot is **Legendrian simple** if for any two Legendrian realisations L_1, L_2

$$\operatorname{tb}(L_1) = \operatorname{tb}(L_2) \text{ and } \operatorname{rot}(L_1) = \operatorname{rot}(L_2) \implies L_1 \text{ and } L_2 \text{ are Legendrian isotopic.}$$

Examples:
Legendrian simple: torus knots, Chekanov twist knots for $m \geq -3$.
Legendrian non-simple: Chekanov twist knots for $m < -3$. [1]
Our aim is to find further examples of Legendrian non-simple knots.

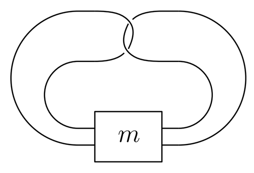

Fig. 3: The Ckekanov twist knot

Two-bridge Knots

Bridge in a knot diagram: an arc between undercrossings that contains at least one overcrossing.
Two-bridge knots have a diagram with exactly two bridges.
Equivalent definition: knots on which the natural height function has only two maxima and two minima as critical points.

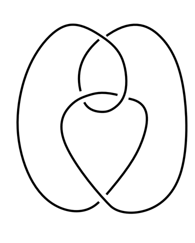
Fig. 4: Two-bridge diagram of the trefoil knot

$$\frac{p}{q} = [a_1, ..., a_{2m+1}] = a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_{2m+1}}}} \longleftrightarrow$$

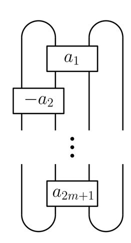
Fig. 5: A general rational knot

Two-bridge knots are also called **rational knots**, since they can be classified with rational numbers.

Schubert's theorem: $K_{\frac{p}{q}}$ is isotopic to $K_{\frac{p'}{q'}} \Leftrightarrow q = q'$ and $p^{\pm 1} \equiv p'$ (mod q).
Double twist knots $K(k, m)$ are two-bridge knots corresponding to continued fractions of the form $[k, m-1, 1]$. They are generalizations of Chekanov twist knots.

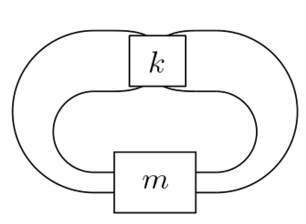

Fig. 6: The double twist knot $K(k, m)$

References

[1] John Etnyre, Lenhard Ng, and Vera Vértesi. "Legendrian and transverse twist knots". In: *Journal of the European Mathematical Society* 15.3 (2013), pp. 969–995. DOI: 10.4171/jems/383.

[2] Viktória Földvári. "Legendrian non-simple two-bridge knots". In: *Periodica Mathematica Hungarica* (Jan. 2019). ISSN: 1588-2829. DOI: 10.1007/s10998-018-00281-y.

[3] Paolo Lisca, Peter Ozsváth, András Stipsicz, and Zoltán Szabó. "Heegard Floer invariants of Legendrian knots in contact three-manifolds". In: *Journal of the European Mathematical Society* 11 (Mar. 2008). DOI: 10.4171/JEMS/183.

[4] Peter Ozsváth and András I. Stipsicz. "Contact surgeries and the transverse invariant in knot Floer homology". In: *Journal of the Institute of Mathematics of Jussieu* 9.3 (2010), pp. 601–632. DOI: 10.1017/S1474748010000095.

[5] *Source of Figure 1:* https://en.wikipedia.org/wiki/Contact_geometry.

Main results

Our contribution includes a lower and an upper bound on the number of Legendrian realizations of some special families of knots, with maximal tb invariant. As corollaries, we obtain new Legendrian non-simple knot types.
Theorem (Lower bound): Suppose that $\frac{p}{q} \in \mathbb{Q}$ has the continued fraction expansion $\frac{p}{q} = [a_1, ..., a_{2m+1}]$ for $m \geq 1$ and suppose that a_2 is odd, a_1 and a_{2i} is even for $i > 1$ and $\sum_{i=1}^{m} a_{2i+1}$ is odd.
Then the corresponding two-bridge knot K admits at least $\lceil \frac{a_1}{4} \rceil$ distinct Legendrian realizations in (S^3, ξ_{std}) with $\operatorname{tb} = \sum_{i=1}^{m} a_{2i+1}$ and $\operatorname{rot} = 0$.
This is a generalization of a theorem by Ozsváth and Stipsicz. [4] The same theorem holds for transverse knots with $\operatorname{sl} = \sum_{i=1}^{m} a_{2i+1}$. [2]
Outline of the proof:
Consider the following Legendrian knot $\widetilde{L}_i \subset S^1 \times D^2$:
Starting with $i = 2$ we put $\widetilde{L}_i$ into the box of $\widetilde{L}_{i-1}$. We define $\widetilde{L}$ as the knot obtained by putting together m pieces this way ($a_{2i} = d_i$, $a_{2i+1} = c_i$ for $i = 1, ..., m$).
Let $U(k, l)$ be the Legendrian realization of the unknot shown in Figure 8.
We can embed $\widetilde{L}$ into a tubular neighborhood of $U(k, l)$, resulting in a new Legendrian knot:
$L_{k,l} = S(U(k, l), \widetilde{L})$, the **Legendrian satellite**. For this, the followings hold ($\operatorname{w}(\widetilde{L}) = 0$ is the **winding number** of $\widetilde{L}$):

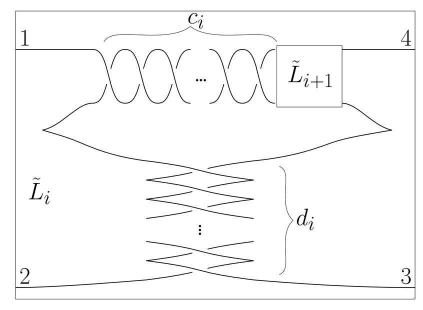

Fig. 7: The solid torus knot $\widetilde{L}_i$.
The data c_i and d_i for $i = 1, ..., m$ determine $\widetilde{L}$

$$\operatorname{tb}(L_{k,l}) = \operatorname{w}(\widetilde{L})^2 \cdot \operatorname{tb}(U(k, l) + \operatorname{tb}(\widetilde{L}) = \sum_{i=1}^{m} c_i$$

$$\operatorname{rot}(L_{k,l}) = \operatorname{w}(\widetilde{L}) \cdot \operatorname{rot}(U(k, l) + \operatorname{rot}(\widetilde{L}) = 0.$$

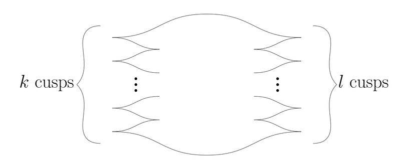

Fig. 8: $U(k, l)$, a Legendrian realization of the unknot

$L_{k,l}$ is smoothly isotopic to the two-bridge knot $K_{\frac{p}{q}}$, where $\frac{p}{q} = [a_1, ..., a_{2m+1}]$.
We show that for different (k, l) pairs, many of the knots $L_{k,l}$ are different Legendrian realizations of $K_{\frac{p}{q}}$.
To this, we use knot Floer homology, and $\mathcal{L}(L) \in HFK^-(-Y, L)$, the **Legendrian invariant** of L [3].
Facts:

- If L_1 and L_2 are Legendrian isotopic Legendrian knots, then there exists an isomorphism from $HFK^-(-Y, L_1)$ to $HFK^-(-Y, L_2)$ mapping $\mathcal{L}(L_1)$ to $\mathcal{L}(L_2)$.
- The Legendrian invariant is an invariant up to automorphisms of $HFK^-(-Y, L)$.
- $\mathcal{L}(L)$ can be lifted from $HFK^-(-Y, L) / Aut(HFK^-(-Y, L))$ to $HFK^-(-Y, L) / \pm MCG(-Y, L)$.

Consider the distinguished triangle of knots on Figure 9. This induces a surgery exact triangle on homologies, see Figure 10.

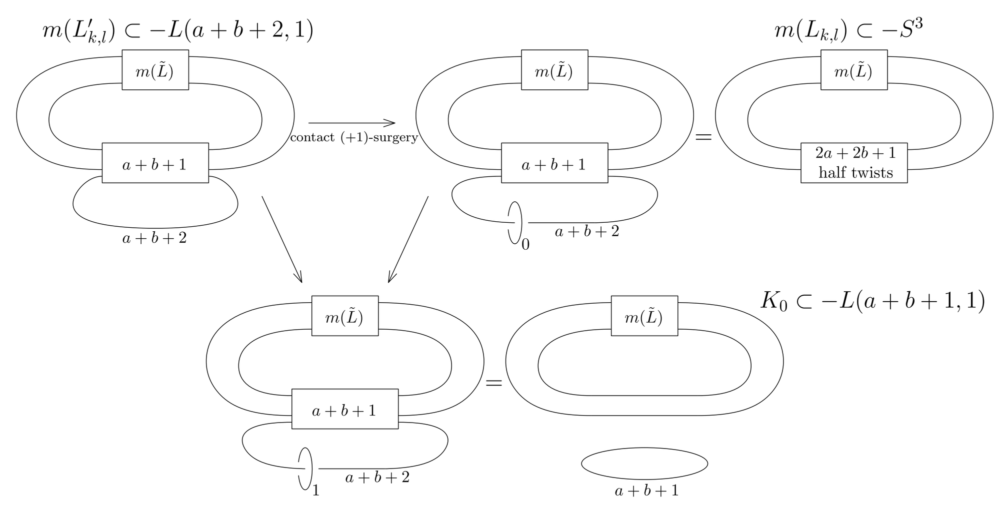
Fig. 9: A distinguished triangle of links

$$\begin{tikzcd}
\widehat{HFK}(-L(a+b+2,1), L'_{k,l}) \arrow[rr, "f"] & & \widehat{HFK}(-S^3, L_{k,l}) \arrow[dl] \\
& \widehat{HFK}(-L(a+b+1,1), m(K_0)) \arrow[ul] &
\end{tikzcd}$$

Fig. 10: The induced surgery exact triangle

The induced maps preserve the Alexander grading, let $\widetilde{A} = A(\mathcal{L}(L_{k,l}))$.
Lemma: When the conditions of the theorem hold, the $\widehat{HFK}$ homology of the third (bottom) term vanishes in Alexander grading $\widetilde{A}$.
From the lemma it follows that the map f is an isomorphism, and it can be shown that for different (k, l) pairs, the Legendrian invariants $\mathcal{L}(L'_{k,l})$ are different.
Counting the number of (k, l) pairs, and considering the action of the mapping class group, we get that the number of different Legendrian realizations of $K_{\frac{p}{q}}$ is at least $\lceil \frac{a_1}{4} \rceil$.
Theorem (Upper bound): For $|m| > 2$, the number of Legendrian realizations of the double twist knot $K(4, m)$ in (S^3, ξ_{std}) with maximal tb is

- at most 10, if $m \geq 2$, and
- at most $10 \cdot \lfloor \frac{(|m|+2)^2}{4} \rfloor$ if $m \leq -2$.

The proof is based on **convex surface theory**. We used technicques apllied in [1], and **ruling invariants**.

Algorithm

The proof of the lower bound theorem has two cardinal points:

- the winding number $\operatorname{w}(\widetilde{L}) = 0$, and
- $\widehat{HFK}(-L(a+b+1,1), K_0) = 0$ in Alexander grading $\widetilde{A}$.

We have algorithmic methods that tell from the continued fraction expansion of any rational number whether these two conditions hold for the corresponding two-bridge knot. This way we get further examples of Legendrian non-simple two-bridge knots, that are not included in the main theorem.
The diagram on the right helps the computation.
Further examples of Legendrian non-simple two-bridge knots include $K_{\frac{153}{179}}$, $K_{\frac{171}{203}}$, $K_{\frac{341}{399}}$.

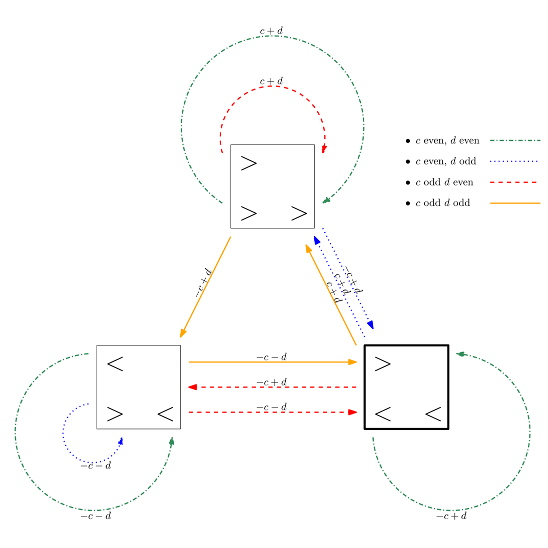
Fig. 11: Computation of the writhe